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An example of convex Hamiltonian diffeomorphism where asymptotic distance from identity is strictly greater than the minimal action

Xiaojun Cui

Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, People's Republic of China

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ABSTRACT

We construct a convex Hamiltonian diffeomorphism on the unit ball of cotangent bundle of $\mathbb{T}^n$ ($n \geqslant 2$), where the asymptotic distance from identity is strictly greater than the minimal action.

1. A brief introduction to Hofer's geometry

In this section, Hofer's geometry is recalled briefly.

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus and $T^*\mathbb{T}^n$ be the cotangent bundle with canonical symplectic form $\omega_0 = d\lambda$, here, and in the following, λ denotes the Liouville form. We use (q, p) to denote the local coordinates of $T^*\mathbb{T}^n$, here $q \in \mathbb{T}^n$ and $p \in T_q^*\mathbb{T}^n$. Hence, λ may be expressed as pdq. We denote the unit ball bundle of $\mathbb{T}^n$ by $B^*\mathbb{T}^n$, that is, $B^*\mathbb{T}^n = \{(q, p) \mid |p| \leqslant 1\}$, here $|\cdot|$ is induced by the standard metric on $\mathbb{T}^n$. We denote by $\mathcal{H}$ the set of all smooth (at least C^2) time 1-periodic Hamiltonians

$$H(q, p, t) : B^*\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$$

(here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the 1-torus) which satisfy the following conditions: for every $H \in \mathcal{H}$,

(1) H vanishes on the boundary of $B^*\mathbb{T}^n$ (denoted by $\partial B^*\mathbb{T}^n$), i.e., $H(q, p, t) = 0$ whenever $|p| = 1$;
(2) H admits a smooth extension $\tilde{H} : T^*\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ which is only a function of t and $|p|^2$ outside $B^*\mathbb{T}^n \times \mathbb{T}$.

In this article, ϕ_H denotes the Hamilton flow generated by Hamiltonian H, and ϕ_H^t denotes the time t-map of the Hamilton flow ϕ_H. If $\phi = \phi_H^1$, then we also say that ϕ is generated by Hamiltonian

E-mail address: xjohncui@gmail.com.

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flow ϕ_H. It follows from the above two conditions that Hamiltonian flow ϕ_H is completeness on $B^*\mathbb{T}^n \times \mathbb{T}$ (i.e., the solutions of Hamilton equation are defined on $\mathbb{R}$) and $\partial B^*\mathbb{T}^n \times \mathbb{T}$ is invariant under the Hamiltonian flow ϕ_H.

Let

$$\mathcal{D} := \{\phi : B^*\mathbb{T}^n \to B^*\mathbb{T}^n \mid \phi = \phi_H^1 \text{ for some } H \in \mathcal{H}\},$$

and $\mathcal{D}$ is called Hamiltonian diffeomorphism group.

A Hamiltonian $H \in \mathcal{H}$ defines a path $t \to \phi_H^t$ in the diffeomorphism group $\mathcal{D}$. In this article, the length of this path is defined by

$$l(H) := \int_{\mathbb{T}} \operatorname{Osc} H_t \, dt,$$

here $H_t = H(\cdot, t) : B^*\mathbb{T} \to \mathbb{R}^n$ and $\operatorname{Osc} H_t := \max H_t - \min H_t$ denotes the oscillation of H_t.

If $\phi \in \mathcal{D}$, set $d(\mathrm{id}, \phi) = \inf\{l(H) \mid H \in \mathcal{H},\ \phi = \phi_H^1\}$ and it is called Hofer distance of ϕ from identity [7]. Given any $H \in \mathcal{H}$, the set

$$\sigma_c(H) := \left\{ \int_{\Gamma} \lambda - H \, dt \;\middle|\; \Gamma \text{ is contractible 1-periodic orbit of the flow } \phi_H \right\}$$

is called the contractible action spectrum of H.

The Hamiltonian is called convex, if $\frac{\partial^2 H}{\partial p^2} > 0$ for any $(q, p, t) \in B^*\mathbb{T}^n \times \mathbb{T}$. If a Hamiltonian diffeomorphism is generated by a convex Hamiltonian, then we say that this Hamiltonian diffeomorphism is convex. The following proposition [13] is a slight generalization of a result [2].

Proposition 1. *Suppose ϕ is generated by a convex Hamiltonian $H \in \mathcal{H}$, then $d(\mathrm{id}, \phi) \geqslant \inf \sigma_c(H) > 0$.*

The asymptotic distance (introduced in [3]) from the identity to an element $\phi \in \mathcal{D}$ is defined as

$$d_\infty(\mathrm{id}, \phi) = \lim_{n \to \infty} \frac{1}{n} d(\mathrm{id}, \phi^n).$$

2. Relation with Mather's minimal action theory

Let $H : B^*\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ be a convex Hamiltonian in $\mathcal{H}$, then it is elementary to show that there exists a smooth convex extension (if H is real analytic, we will use C^∞ extensions) $\tilde{H} : T^*\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ of H such that $\tilde{H}$ is a function of t and $|p|^2$ outside $B^*\mathbb{T}^n$ and has superlinear growth (i.e., for any $(q, t) \in \mathbb{T}^n \times \mathbb{T}$, $\frac{\tilde{H}(q,p,t)}{|p|} \to +\infty$ as $|p| \to +\infty$). Now we consider the related (by Legendre transformation) Lagrangian $\tilde{L} : T\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ (here $T\mathbb{T}^n$ denotes the tangent bundle of $\mathbb{T}^n$, and we use $(q, \dot{q})$ to denote the local coordinates, $q \in \mathbb{T}^n$ and $\dot{q} \in T_q\mathbb{T}^n$), i.e.,

$$\tilde{L}(q, \dot{q}, t) = p\dot{q} - \tilde{H}, \quad \dot{q} = \frac{\partial \tilde{H}}{\partial \dot{q}}.$$

Hence, Mather's α-function and β-function [10] (both two functions are called Mather's minimal action functions) is well defined for Lagrangian $\tilde{L}$. More precisely,

$$\beta_{\tilde{L}}(\omega) = \min_{[\mu]=\omega} \int \tilde{L} \, d\mu,$$

here μ ranges over all invariant (under the flow $\phi_{\tilde{L}}$, the Euler–Lagrange flow associated to Lagrangian $\tilde{L}$) Borel probability measure in the extended phase space $T\mathbb{T}^n \times \mathbb{T}$ with $\int \tilde{L}\,d\mu < +\infty$, $[\mu] \in H_1(\mathbb{T}^n, \mathbb{R})$ denotes the rotation vector of μ defined by

$$\langle [\mu], [\eta]_{\mathbb{T}^n} \rangle + [\eta]_{\mathbb{T}} = \int \eta\, d\mu$$

for any closed 1-form η on $\mathbb{T}^n \times \mathbb{T}$, here

$$[\eta] = \left([\eta]_{\mathbb{T}^n}, [\eta]_{\mathbb{T}}\right) \in H^1\left(\mathbb{T}^n \times \mathbb{T}, \mathbb{R}\right) = H^1\left(\mathbb{T}^n, \mathbb{R}\right) \times \mathbb{R}$$

denotes the first de Rham cohomology class of η, and $\langle \cdot,\cdot \rangle$ denotes the canonical pair between the cohomology and the homology. The convex conjugate of the β-function is called α-function, i.e., α is a function on $H^1(\mathbb{T}^n, \mathbb{R})$ defined by

$$\alpha_{\tilde{L}}(c) = -\min_{\omega \in H_1(\mathbb{T}^n, \mathbb{R})} \left(\beta_{\tilde{L}}(\omega) - \langle c, \omega \rangle\right).$$

Since $\tilde{L}$ is uniquely determined by $\tilde{H}$, we may also denote these two functions by $\beta_{\tilde{H}}(\omega)$ and $\alpha_{\tilde{H}}(c)$ for convenience. Siburg [12,13] proved the following

Proposition 2. *The value $\beta_{\tilde{H}}(0)$ is independent of the particular extension $\tilde{H}$.*

Hence, the value $\beta_H(0)$ is well defined. Moreover, Siburg [12,13] also proved the following

Proposition 3. *Suppose $\phi \in \mathcal{D}$ is generated by a convex Hamiltonian $H \in \mathcal{H}$. Then $d_\infty(\mathrm{id}, \phi) \geqslant \beta_H(0)$.*

Based on this proposition, Siburg [13] posed the following open problem:

Open problem: *Whether is it true that $d_\infty(\mathrm{id}, \phi) = \beta_H(0)$?*

In this article, we will construct an example of convex Hamiltonian $H \in \mathcal{H}$, where $d_\infty(\mathrm{id}, \phi_H^1) > \beta_H(0)$. Hence, the problem is answered negatively.

To state our example, we need the notion of Aubry set, which is introduced by Mather [10,11]. For Lagrangian $\tilde{L}$ (which satisfies Mather's conditions: positive definiteness, superlinear growth, and completeness [10,11]), let

$$h\big((q_1, \tau_1), (q_2, \tau_2)\big) = \lim_{T \to \infty} \inf_{\gamma} \int \big(\tilde{L} + \alpha_{\tilde{L}}(0)\big)\big(\gamma(t), \dot{\gamma}(t), t \mod 1\big)\, dt,$$

where the infimum is taken over all absolutely continuous curves with $\gamma(t_1) = q_1, \gamma(t_2) = q_2$ and $t_1 = \tau_1 (\mod 1), t_2 = \tau_2 (\mod 1)$, $t_2 - t_1 \geqslant T$. Let

$$\rho\big((q_1, \tau_1), (q_2, \tau_2)\big) = h\big((q_1, \tau_1), (q_2, \tau_2)\big) + h\big((q_2, \tau_2), (q_1, \tau_1)\big).$$

Let $\mathcal{A} = \{(q, \tau) \in \mathbb{T}^n \times \mathbb{T} \mid \rho((q, \tau), (q, \tau)) = 0\}$. In Mather's theory, $\mathcal{A}$ is called projected Aubry set (associated to cohomology class 0). Let $\dot{\mathcal{A}}$ denote the Aubry set, i.e.,

$$\dot{\mathcal{A}} = \left\{(q, \dot{q}, \tau)\colon \int_{t_1}^{t_2} \big(\tilde{L} + \alpha(0)\big)\phi_{\tilde{L}}^t(q, \dot{q}, \tau)\, dt = -h\big((q_2, \tau_2), (q_1, \tau_1)\big) \text{ for any } t_1 \leqslant t_2 \in \mathbb{R}\right\},$$

here $\phi_{\tilde{L}}^t$ denotes the Euler–Lagrange flow associated to Lagrangian $\tilde{L}$, $\pi \circ \phi_{\tilde{L}}^{t_1}(q, \dot{q}, \tau) = (q_1, \tau_1)$ and $\pi \circ \phi_{\tilde{L}}^{t_2}(q, \dot{q}, \tau) = (q_2, \tau_2)$ (here π denotes the canonical projection of $T\mathbb{T}^n \times \mathbb{T} \to \mathbb{T}^n \times \mathbb{T}$). Then $\pi : \dot{\mathcal{A}} \to \mathcal{A}$ is a bi-Lipschitz homeomorphism.

3. The example

The configuration space we consider is the n-torus $\mathbb{T}^n$ ($n \geqslant 2$), endowed with the standard metric. Let V be a C^∞ vector field on $\mathbb{T}^n$. We assume that there exist two non-contractible closed orbits $\gamma_1(t)$ and $\gamma_2(t)$ for the flow ϕ_V generated by vector field V. Moreover, we assume that their rotation vectors are converse, that is, $\int_{\gamma_1} \eta' = -\int_{\gamma_2} \eta'$ for any closed 1-form η' on $\mathbb{T}^n$.

Now we consider the Lagrangian $L_0 : T\mathbb{T}^n \times \mathbb{T} (= \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}) \to \mathbb{R}$ of the following type:

$$L_0(q, \dot{q}, t) = \frac{1}{2}\left|\dot{q} - \frac{1}{k}V(q)\right|^2 + U(q,t) + \frac{1}{2},$$

here $q \in \mathbb{T}^n$, $\dot{q} \in T_q\mathbb{T}^n$, the positive integer k and the nonnegative potential function $U(q,t)$ are to be determined.

Obviously, the related (by Legendre transformation) Hamiltonian has the following form:

$$H_0(q, p, t) = \frac{1}{2}\left(|p|^2 - 1\right) + \left\langle p, \frac{1}{k}V(q)\right\rangle - U(q,t),$$

here $p = \frac{\partial L_0}{\partial \dot{q}} = \dot{q} - \frac{1}{k}V(q)$.

Now we modify Hamiltonian H_0 to $\tilde{H}$ in the following way:

$$\tilde{H}(q, p, t) = \frac{1}{2}\left(|p|^2 - 1\right) + f\left(|p|^2\right)\left(\left\langle p, \frac{1}{k}V(q)\right\rangle - U(q,t)\right),$$

here $f : [0, \infty) \to [0, 1]$ is a smooth function with $f(s) = 1$ if $s \leqslant \frac{1}{2}$ and $f(s) = 0$ if $s \geqslant 1$.

By the definition of $\tilde{H}$, to ensure that $\tilde{H}$ is convex with respect to the variable p, the following two hypotheses are enough:

1. $k \in \mathbb{Z}^+$ is sufficiently large.
2. $\max U(q,t)$ is sufficiently small (recall that we have assumed that $U(q,t) \geqslant 0$).

In the following of this paper, we assume that these two hypotheses are satisfied and so $\tilde{H}$ is a convex (with respect to the variable p) Hamiltonian on $T^*\mathbb{T}^n \times \mathbb{T}$. Note that $\tilde{H}$ has superlinear growth. Moreover, $\tilde{H}(q,p,t) = 0$ when $|p| = 1$. Note that $B^*\mathbb{T}^n \times \mathbb{T}$ is an invariant set under the flow $\phi_{\tilde{H}}$. Let $H(q,p,t) = \tilde{H}(q,p,t)|_{B^*\mathbb{T}^n \times \mathbb{T}}$. Obviously, $H \in \mathcal{H}_0$, since H admits a smooth, convex extension $\tilde{H} : T^*\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ which is only a function of $|p|^2$ outside $B^*\mathbb{T}^n \times \mathbb{T}$.

In our example, there is an extra hypothesis:

3. The function $U : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ satisfies: $U = 0$ on the set

$$\left(\gamma_1(t), t \mod 1\right) \cup \left(\gamma_2(t), t \mod 1\right)$$

   and $U > 0$ otherwise.

Clearly, the projected Aubry set (associated to the 0-cohomology class) $\mathcal{A}$ of L_0 (or equivalently, H_0) is exactly the union of the two closed orbits, i.e., the set

$$(q,t) \in \left(\gamma_1(t), t \mod 1\right) \cup \left(\gamma_2(t), t \mod 1\right).$$

By a time re-parameterization, i.e., $s = kt$, we have

$$\dot{\mathcal{A}} = \left(\gamma_1(s), \dot{\gamma}_1(s), s \mod 1\right) \cup \left(\gamma_2(s), \dot{\gamma}_2(s), s \mod 1\right),$$

here $\dot{} = \frac{d}{ds}$. In the following, γ_1 and γ_2 are parameterized by parameter s and we still denotes the variable s by t for convenience. Note that $\dot{\mathcal{A}}$ supports an invariant probability measure (the so-called minimal measure [10]) with 0-rotation vector, since rotation vectors of γ_1 and γ_2 are converse. Hence, $\beta_{H_0}(0) = -\alpha_{H_0}(0) = \frac{1}{2}$, here the function β_{H_0} denotes the β-function associated to the Hamiltonian H_0 (or equivalently, associated to the Lagrangian L_0).

Note that the Hamiltonian $\tilde{H}$ is convex and has superlinear growth, so the Mather's β-function $\beta_{\tilde{H}}$ is also well defined. Moreover, we have

Proposition 4. $-\alpha_{\tilde{H}}(0) = \beta_{\tilde{H}}(0) = \frac{1}{2}$.

Before entering into the proof, we recall some fundamental results on viscosity subsolutions of Hamilton–Jacobi equations. For a Hamiltonian $H : T^*\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$, we have the associated Hamilton–Jacobi equations

$$\partial_t u + H(q, \partial_q u, t) = d, \tag{$*$}$$

here d is any real constant. A function $u : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ is called a viscosity subsolution of Hamilton–Jacobi equation ($*$) if u satisfies the following condition: for every C^1 function $\phi : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ and every point $(q,t) \in \mathbb{T}^n \times \mathbb{T}$ such that $u - \phi$ has a maximum at (q,t), we have $\partial_t\phi + H(q, \partial_q\phi, t) \leqslant d$. Clearly, a C^1 function u is a viscosity subsolution if and only if $\partial_t\phi + H(q, \partial_q\phi, t) \leqslant d$ for any point $(q,t) \in \mathbb{T}^n \times \mathbb{T}$. By weak KAM theory [4,6,8], viscosity subsolutions of Hamilton–Jacobi equation are closely related to minimal orbits in Mather's theory when the associated Hamiltonian is convex and has superlinear growth.

Proof of Proposition 4. Note that $u = 0$ is a viscosity subsolution of the Hamilton–Jacobi equation

$$\partial_t u + H_0(q, \partial_q u, t) = -\frac{1}{2}.$$

Since $\tilde{H}(q,p,t) = H_0(q,p,t)$ when $|p| \leqslant \frac{1}{2}$, $u = 0$ is also a viscosity subsolution of the Hamilton–Jacobi equation

$$\partial_t u + \tilde{H}(q, \partial_q u, t) = -\frac{1}{2}.$$

So $\beta_{\tilde{H}}(0) = -\min_{c \in H^1(\mathbb{T}^n,\mathbb{R})} \alpha(c) \geqslant -\alpha(0) \geqslant \frac{1}{2}$, since $\alpha(0)$ is infimum of the set of constants d with

$$\partial_t u + \tilde{H}(q, \partial_q u, t) = d$$

has viscosity subsolutions, by the weak KAM theory of time-periodic case [4,8]. On the other hand, $(\gamma_1(t), \dot{\gamma}_1(t), t \mod 1)$ and $(\gamma_2(t), \dot{\gamma}_2(t), t \mod 1)$ are two closed orbits of $\tilde{L}$, here $\tilde{L}$ and $\tilde{H}$ are related by Legendre transformation. We denote the invariant probability measure supported on $(\gamma_1(t), \dot{\gamma}_1(t), t \mod 1)$ by μ_1, and the invariant probability measure supported on $(\gamma_2(t), \dot{\gamma}_2(t), t \mod 1)$ by μ_2. Let $\mu_0 = \frac{1}{2}(\mu_1 + \mu_2)$. Obviously, the rotation vector of μ_0 is zero. Clearly, we have $\int \tilde{L}\, d\mu_0 = \frac{1}{2}$, since $\tilde{L} = L_0$ when it is restricted on the union of $(\gamma_1(t), \dot{\gamma}_1(t), t \mod 1)$ and $(\gamma_2(t), \dot{\gamma}_2(t), t \mod 1)$. Hence $\beta_{\tilde{H}}(0) = \frac{1}{2}$. Moreover, by the fact that $u = 0$ is also a viscosity subsolution of the Hamilton–Jacobi equation

$$\partial_t u + \tilde{H}(q, \partial_q u, t) = -\frac{1}{2},$$

it is also easy to see that $\alpha_{\tilde{H}}(0) = \min_{c \in H^1(\mathbb{T}^n,\mathbb{R})} \alpha_{\tilde{H}}(c) = -\frac{1}{2}$ in our case. $\square$

Note that the value $\beta_{\tilde{H}}(0) = -\min_{c\in H^1(\mathbb{T}^n,\mathbb{R})}\alpha(c)$ is independent of the choice of the extension, so we may say that this value depends only on the Hamilton $H \in \mathcal{H}$ and denoted by $\beta_H(0)$.

For a closed orbit $\Gamma(t) = (\gamma(t), p(t))$ of the flow $\phi_{\tilde{H}}$, we denote the action $\int_\Gamma \lambda - \tilde{H}\,dt$ by $A(\Gamma)$. Since $\tilde{H}$ is convex and has superlinear growth, we have that $(\gamma(t), \dot{\gamma}(t), t \mod 1)$ is a closed orbit of the Euler–Lagrange flow $\phi_{\tilde{L}}$ and

$$A(\Gamma) = \int \tilde{L}\big(\gamma(t), \dot{\gamma}(t), t \mod 1\big)\,dt \;\big(\triangleq A(\gamma)\big).$$

Denote $\inf_\Gamma A(\Gamma)$ by $A(n)$, here Γ ranges over all contractible n-periodic orbit of the flow $\phi_{\tilde{H}}$. Obviously, if we let Γ ranges over all contractible n-periodic orbit of the flow ϕ_H in the definition of $A(n)$, we get the same infimum.

Proposition 5. *If ϕ is generated by a convex Hamiltonian $H \in \mathcal{H}$, then*

$$d_\infty(\mathrm{id}, \phi) \geqslant \inf_{n\in\mathbb{Z}^+} \frac{1}{n} A(n).$$

Proof. By the definition, $d_\infty(\mathrm{id}, \phi) = \lim_{n\to\infty} \frac{1}{n} d(\mathrm{id}, \phi^n)$. Note that ϕ^n may be generated (as the time 1-map) by the Hamiltonian

$$H_{\#n}(q, p, t) = nH(q, p, nt).$$

By Proposition 1, $d(\mathrm{id}, \phi^n) \geqslant \inf \sigma_c(H_{\#n})$. Note that $\Gamma(t) = (\gamma(t), p(t))$ is a contractible n-periodic orbit of Hamilton flow ϕ_H if and only if $\Gamma^*(t) = (\gamma(nt), p(nt))$ is a contractible 1-periodic orbit of Hamilton flow $\phi_{H_{\#n}}$. Note that

$$\begin{aligned}
A_{\#n}(\Gamma^*) &= \int_{\Gamma^*} \lambda - H_{\#n}\,dt \\
&= \int_0^1 \big(p(nt)\dot{\gamma}(nt) - H_{\#n}\big(\gamma(nt), p(nt), t\big)\big)\,dt \\
&= \int_0^n \big(p(t)\dot{\gamma}(t) - H\big(q(t), \gamma(t), t \mod 1\big)\big)\,dt \\
&= \int_\Gamma \lambda - H\,dt \\
&= A(\Gamma),
\end{aligned}$$

here $\dot{} = \frac{d}{dt}$. Hence, we have

$$\sigma_c(H_{\#n}) = \big\{A(\Gamma)\colon\ \Gamma \text{ is contractible } n\text{-periodic orbit of } H\big\}.$$

It follows that

$$\frac{1}{n} d\big(\mathrm{id}, \phi^n\big) \geqslant \frac{1}{n} \inf \sigma_c(H_{\#n}) \geqslant \frac{1}{n} A(n).$$

Note that $\frac{1}{n} d(\mathrm{id}, \phi^n)$ is a non-increasing sequence, so the proposition follows. $\square$

In the following, we will show that

Proposition 6. $\inf_{n\in\mathbb{Z}^+} \frac{1}{n}A(n) > \frac{1}{2} = \beta_H(0)$.

For a closed orbit $(\gamma(t), \dot\gamma(t), t \mod 1)$ of Euler–Lagrange flow $\phi_{\tilde L}$, we denote the action

$$\int \left(\tilde L - \frac{1}{2}\right)\big(\gamma(t), \dot\gamma(t), t\big)\, dt$$

by $A'(\xi)$. Similarly, we define $A'(n) = \inf_\gamma A'(\gamma)$, here γ ranges over all C^1 closed curves such that $(\gamma(t), \dot\gamma(t), t \mod 1)$ is a contractible n-periodic orbit of Euler–Lagrange flow $\phi_{\tilde L}$. To prove Proposition 6, it is enough to show that

Proposition 7.

$$\inf_{n\in\mathbb{Z}^+} \frac{1}{n}A'(n) > 0.$$

Note that in our case, $\beta(0) = -\alpha(0) = \frac{1}{2}$. By Proposition 6, together with Proposition 5, we know that the asymptotic distance from identity is strictly greater than $\beta(0)$. Hence we need only to prove Proposition 7.

Proof of Proposition 7. Clearly, for any contractible periodic orbit $(\gamma(t), \dot\gamma(t), t \mod 1)$ of the Euler–Lagrange flow $\phi_{\tilde L}$, we have $A'(\gamma) > 0$, since projected Aubry set

$$\mathcal{A} = \big(\gamma_1(t), t \mod 1\big) \cup \big(\gamma_2(t), t \mod 1\big)$$

and γ_1, γ_2 are not contractible.

So, if $\inf_{n\in\mathbb{Z}^+} \frac{1}{n}A'(n) = 0$, then there exists a sequence of contractible periodic orbits

$$\big(\xi_i(t), \dot\xi_i(t), t \mod 1\big)$$

with period n_i ($i \to \infty$ when $n_i \to \infty$) such that $\frac{1}{n_i}A'(\xi_i) \to 0$. It is known that

$$\bigcup \big(\xi_i(t), \dot\xi_i(t), t \mod 1\big)$$

is relatively compact in $T\mathbb{T}^n \times \mathbb{T}$, since $\tilde L$ has superlinear growth. So by passing through the subsequence if necessary, we may assume that the sequence $(\xi_i(t), \dot\xi_i(t), t \mod 1)$ is convergent in the Hausdorff topology. We denote the limit by $\dot{\mathcal{L}}$. Hence, $\dot{\mathcal{L}} \subset \dot{\mathcal{A}}$, here $\dot{\mathcal{A}}$ is the Aubry set associated to the cohomology class 0, since $\alpha(0) = -\frac{1}{2}$. Moreover, $\dot{\mathcal{L}}$ is connected and invariant under the flow $\phi_{\tilde L}$, since $(\xi_i(t), \dot\xi_i(t), t \mod 1)$ is connected and invariant under the flow $\phi_{\tilde L}$ for each i. Note that $(\xi_i(t), \dot\xi_i(t), t \mod 1)$ is contractible for each i, the rotation vector of the invariant probability measure supported on $(\xi_i(t), \dot\xi_i(t), t \mod 1)$ is zero. Hence the limit measure should also have the rotation vector zero, so

$$\dot{\mathcal{L}} = \big(\gamma_1(t), \dot\gamma_1(t), t \mod 1\big) \cup \big(\gamma_2(t), \dot\gamma_2(t), t \mod 1\big).$$

But it contradicts to the fact that $\dot{\mathcal{L}}$ is connected. The contradiction shows that

$$\inf_{n\in\mathbb{Z}^+} \frac{1}{n}A'(n) > 0. \qquad \square$$

Remark. There is an example [5] similar to ours, but the purpose is very different.

4. A generalization

By a slight generalization, we can answer the following problem negatively:

Problem. For any convex Hamiltonian diffeomorphism $\phi \in \mathcal{D}$, is it true that

$$d_\infty(\mathrm{id}, \phi) = \sup_H \beta_H(0),$$

where H ranges over all convex Hamiltonians in $\mathcal{H}$ such that $\phi_H^1 = \phi$?

Proposition 8. *There exists a Hamiltonian diffeomorphism $\phi \in \mathcal{D}$, such that*

$$d_\infty(\mathrm{id}, \phi) > \sup_H \beta_H(0),$$

where H ranges over all convex Hamiltonians in $\mathcal{H}$ such that $\phi_H^1 = \phi$.

To prove this proposition, we only need to show that

Lemma. *Let $H_1, H_2 \in \mathcal{H}$ be any two convex Hamiltonians with $\phi_{H_1}^1 = \phi_{H_2}^1 = \phi$, then $\beta_{H_1}(0) = \beta_{H_2}(0)$.*

Proof. For any invariant (under the Hamiltonian diffeomorphism ϕ) Borel probability measure μ on $B^*\mathbb{T}^n$, let $A_H(\mu) = \int_{B^*\mathbb{T}^n} d\mu(q,p) \int_{\Gamma_{q,p}} \lambda - H\,dt$, here $\Gamma(q,p)$ is the Hamiltonian trajectory generated by Hamiltonian flow ϕ_H between $t = 0$ and $t = 1$, starting at (q,p). By a nice exposition [9],

$$A_{H_1}(\mu) - A_{H_2}(\mu) = C$$

for any invariant measure of μ (with respect to the Hamiltonian diffeomorphism ϕ), here H_1 and H_2 are two convex Hamiltonians in $\mathcal{H}$ and $\phi_{H_1}^1 = \phi_{H_2}^1 = \phi$, C is a constant which is independent of invariant measure μ. Note that the boundary of $B^*\mathbb{T}^n$ (i.e., the set $\{(q,p)\colon |p| = 1\}$) is a compact invariant subset under the diffeomorphism ϕ. We recall that H vanishes on the boundary of $B^*\mathbb{T}^n$, for any Hamiltonian $H \in \mathcal{H}$. Hence, for any invariant measure (under the Hamiltonian diffeomorphism ϕ) supported on the boundary of $B^*\mathbb{T}^n$, we have $A_{H_1}(\mu) = A_{H_2}(\mu)$ for any Hamiltonians $H_1, H_2 \in \mathcal{H}$ and $\phi_{H_1}^1 = \phi_{H_2}^2 = \phi$. So the constant C in the above formula must be zero and consequently, $A_{H_1}(\mu) = A_{H_2}(\mu)$ for any ϕ-invariant measure μ. By the definition of β function, we know that $\beta_{H_1}(0) = \beta_{H_2}(0)$ in the case that H_1, H_2 are convex Hamiltonians in $\mathcal{H}$ and $\phi_{H_1}^1 = \phi_{H_2}^1 = \phi$. $\square$

This lemma, together with the example in Section 3, shows that Proposition 8 is true.

Remark. When I submitted this paper, I was not aware of whether Proposition 8 is true. After the submission, I realized that $\beta_H(0)$ is independent of Hamiltonian H such that $\phi_H^1 = \phi$, by the above method. It follows that Proposition 8 holds. An anonymous referee also pointed out that this fact follows from symplectic homogenization theory of Viterbo [14] and the work of Bernard [1]. The referee also pointed out that Hofer's distance may not be the right thing to look at if one wants an equality and Viterbo's distance might do the trick.

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References

[1] P. Bernard, Symplectic aspects of Aubry–Mather theory, Duke Math. J. 136 (3) (2007) 401–420.
[2] M. Bialy, L. Polterovich, Geodesic of Hofer's metric on the group of Hamiltonian diffeomorphisms, Duke Math. J. 76 (1994) 273–292.
[3] M. Bialy, L. Polterovich, Invariant tori and symplectic topology, Amer. Math. Soc. Transl. 171 (1996) 23–33.
[4] G. Contreras, R. Iturriaga, H. Sanchez-Morgado, Weak solutions of the Hamilton–Jacobi equations for time periodic Lagrangians, preprint.
[5] G. Contreras, L. Macarini, G.P. Paternain, Periodic orbits for exact magnetic flows on surfaces, Int. Math. Res. Not. 8 (2004) 361–387.
[6] A. Fathi, Weak KAM theorem in Lagrangian dynamics, seventh preliminary version, 2005.
[7] H. Hofer, E. Zehnder, Symplectic Invariants and Hamiltonian Dynamics, Birkhäuser, 1994.
[8] D. Massart, Subsolutions of time-periodic Hamilton–Jacobi equations, preprint, 2007.
[9] J. Mather, Minimal measures, Comment. Math. Helv. 64 (1989) 375–394.
[10] J. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z. 207 (1991) 169–207.
[11] J. Mather, Total disconnectedness of the quotient Aubry set in low dimensions, Comm. Pure Appl. Math. 56 (2003) 1178–1183.
[12] K. Siburg, Action-minimizing measures and the geometry of the Hamiltonian diffeomorphism group, Duke Math. J. 92 (1998) 295–319.
[13] K. Siburg, The Principle of Least Action in Geometry and Dynamics, Lecture Notes in Math., vol. 1844, 2004.
[14] C. Viterbo, Symplectic homogenization, preprint, 2008, 55 pp.